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Research Article

Fixed Point Iterations of a Pair of Hemirelatively Nonexpansive Mappings

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We introduce an iterative method for a pair of hemirelatively nonexpansive mappings. Strong convergence of the purposed iterative method is obtained in a Banach space.

1. Introduction and Preliminaries

Let E be a Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. A Banach space E is said to be strictly convex if $\|(x+y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|(x_n + y_n)/2\| = 1$. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided that

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.2}$$

exists for each $x, y \in U_E$. It is also said to be uniformly smooth if the limit (1.2) is attained uniformly for $x, y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that E is uniformly smooth if and only if E^* is uniformly convex.

Recall that a Banach space E has the Kadec-Klee property if for any sequences $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$; for more details on Kadec-Klee property, the readers is referred to [1, 2] and the references therein. It is well known that if E is a uniformly convex Banach space, then E enjoys the Kadec-Klee property.

Let C be a nonempty closed and convex subset of a Banach space E and $T : C \to C$ a mapping. The mapping T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$. A point $x \in C$ is a fixed point of T provided $Tx = x$. In this paper, we use $F(T)$ to denote the fixed point set of T and use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

Recall that the mapping T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.3}$$

It is well known that if C is a nonempty bounded closed and convex subset of a uniformly convex Banach space E, then every nonexpansive self-mapping T on C has a fixed point. Further, the fixed point set of T is closed and convex.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [3] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for } x, y \in E. \tag{1.4}$$

Observe that, in a Hilbert space H, (1.4) is reduced to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$, the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y\in C} \phi(y, x). \tag{1.5}$$

Existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, e.g., [1–4]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{1.6}$$

Remark 1.1. If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From

(1.6), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, we have $Jx = Jy$. Therefore, we have $x = y$; see [1, 2] for more details.

Let C be a nonempty closed convex subset of E and T a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T [5] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is said to be relatively nonexpansive [3, 6, 7] if $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The mapping T is said to be hemirelatively nonexpansive [8–12] if $F(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mappings was studied in [3, 6, 7].

Remark 1.2. The class of hemirelatively nonexpansive mappings is more general than the class of relatively nonexpansive mappings which requires the restriction: $F(T) = \widetilde{F}(T)$. From Su et al. [11], we see that every hemirelatively nonexpansive mapping is relatively nonexpansive, but the inverse is not true. Hemirelatively nonexpansive mapping is also said to be quasi-ϕ-nonexpansive; see [13–17].

Recently, fixed point iterations of relatively nonexpansive mappings and hemirelatively nonexpansive mappings have been considered by many authors; see, for example [14–25] and the references therein. In 2005, Matsushita and Takahashi [8] considered fixed point problems of a single relatively nonexpansive mapping in a Banach space. To be more precise, they proved the following theorem.

Theorem MT. *Let E be a uniformly convex and uniformly smooth Banach space; let C be a nonempty closed convex subset of E; let T be a relatively nonexpansive mapping from C into itself; let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$
\begin{aligned}
x_0 &= x \in C,\\
y_n &= J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n),\\
H_n &= \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\},\\
W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\},\\
x_{n+1} &= P_{H_n\cap W_n}x_0, \quad n = 0, 1, 2, \ldots,
\end{aligned}
\tag{1.7}
$$

where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the generalized projection from C onto $F(T)$.

In 2007, Plubtieng and Ungchittrakool [9] further improved Theorem MT by considering a pair of relatively nonexpansive mappings. To be more precise, they proved the following theorem.

Theorem PU. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let S and T be two relatively nonexpansive mappings from C into itself with $F := F(T) \cap F(S)$ being nonempty. Let a sequence $\{x_n\}$ be defined by*

$$
\begin{aligned}
x_0 &= x \in C,\\
y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n),\\
z_n &= J^{-1}\left(\beta_n^1 J x_n + \beta_n^2 J T x_n + \beta_n^3 J S x_n\right),\\
H_n &= \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\},\\
W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\},\\
x_{n+1} &= \Pi_{H_n \cap W_n} x, \quad \forall n \ge 0,
\end{aligned}
\tag{1.8}
$$

with the following restrictions:

(1) $0 \le \alpha_n < 1$ *for each* $n \ge 0$ *and* $\limsup_{n\to\infty} \alpha_n < 1$;

(2) $0 \le \beta_n^1, \beta_n^2, \beta_n^3 \le 1$, $\beta_n^1+\beta_n^2+\beta_n^3 = 1$ *for each* $n \ge 0$, $\lim_{n\to\infty} \beta_n^1 = 0$ *and* $\liminf_{n\to\infty} \beta_n^2\beta_n^3 > 0$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x$, where Π_F is the generalized projection from C onto F.

Very recently, Su et al. [11] improved Theorem PU partially by considering a pair of hemirelatively nonexpansive mappings. To be more precise, they obtained the following results.

Theorem SWX. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let S and T be two closed hemirelatively nonexpansive mappings from C into itself with $F := F(T) \cap F(S)$ being nonempty. Let a sequence $\{x_n\}$ be defined by*

$$
\begin{aligned}
x_0 &= x \in C,\\
y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n),\\
z_n &= J^{-1}\left(\beta_n^1 J x_n + \beta_n^2 J T x_n + \beta_n^3 J S x_n\right),\\
C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \le \phi(z, x_n)\},\\
C_0 &= \{z \in C \cap Q_{n-1} : \phi(z, y_0) \le \phi(z, x_0)\},\\
Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx - Jx_n\rangle \ge 0\},\\
Q_0 &= C,\\
x_{n+1} &= \Pi_{C_n \cap Q_n} x_0, \quad \forall n \ge 0,
\end{aligned}
\tag{1.9}
$$

with the following restrictions:

(1) $\liminf_{n\to\infty} \beta_n^1\beta_n^2 > 0$;

(2) $\liminf_{n\to\infty} \beta_n^1\beta_n^3 > 0$;

(3) $0 \le \alpha_n \le \alpha < 1$ *for some* $\alpha \in (0,1)$.

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_F x$, *where* Π_F *is the generalized projection from* C *onto* F.

In this paper, motivated by Theorems MT, PU, and SWX, we consider the problem of finding a common fixed point of a pair of hemirelatively nonexpansive mappings by shrinking projection methods which were introduced by Takahashi et al. [26] in Hilbert spaces. Strong convergence theorems of common fixed points are established in a Banach space. The results presented in this paper mainly improve the corresponding results announced in Matsushita and Takahashi [8], Nakajo and Takahahsi [27], and Su et al. [11].

In order to prove our main results, we need the following lemmas.

Lemma 1.3 (see [3]). *Let* C *be a nonempty closed convex subset of a smooth Banach space* E *and* $x \in E$. *Then,* $x_0 = \Pi_C x$ *if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \ge 0 \quad \forall y \in C. \tag{1.10}$$

Lemma 1.4 (see [3]). *Let* E *be a reflexive, strictly convex and smooth Banach space,* C *a nonempty closed convex subset of* E, *and* $x \in E$. *Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x) \quad \forall y \in C. \tag{1.11}$$

The following lemma can be deduced from Matsushita and Takahashi [8].

Lemma 1.5. *Let* E *be a strictly convex and smooth Banach space,* C *a nonempty closed convex subset of* E *and* $T : C \to C$ *a hemirelatively nonexpansive mapping. Then* $F(T)$ *is a closed convex subset of* C.

Lemma 1.6 (see [28]). *Let* E *be a uniformly convex Banach space and* $B_r(0)$ *a closed ball of* E. *Then there exists a continuous strictly increasing convex function* $g : [0,\infty) \to [0,\infty)$ *with* $g(0) = 0$ *such that*

$$\left\|\lambda x + \mu y + \gamma z\right\|^2 \le \lambda\|x\|^2 + \mu\left\|y\right\|^2 + \gamma\|z\|^2 - \lambda\mu g\left(\left\|x - y\right\|\right) \tag{1.12}$$

for all $x, y, z \in B_r(0)$ *and* $\lambda, \mu, \gamma \in [0,1]$ *with* $\lambda + \mu + \gamma = 1$.

2. Main Results

Theorem 2.1. *Let* E *be a uniformly smooth and strictly convex Banach space which enjoys the Kadec-Klee property and* C *a nonempty closed and convex subset of* E. *Let* $T : C \to C$ *and* $S : C \to C$ *be* *two closed and hemirelatively nonexpansive mappings such that $\mathcal{F} = F(T) \cap F(S)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$
\begin{gathered}
x_0 \in E \text{ chosen arbitrarily}, \\
C_1 = C, \\
x_1 = \Pi_{C_1} x_0, \\
z_n = J^{-1}(\beta_{n,0} J x_n + \beta_{n,1} J T x_n + \beta_{n,2} J S x_n), \\
y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n), \\
C_{n+1} = \{ z \in C_n : \phi(z, y_n) \le \phi(z, x_n) \}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 0,
\end{gathered}
\tag{2.1}
$$

where $\{\alpha_n\}$, $\{\beta_{n,0}\}$, $\{\beta_{n,1}\}$, and $\{\beta_{n,2}\}$ are real sequences in $[0,1]$ satisfying the following restrictions:

(a) $\limsup_{n\to\infty} \alpha_n < 1$;

(b) $\beta_{n,0} + \beta_{n,1} + \beta_{n,2} = 1$;

(c) $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,1} > 0$ *and* $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,2} > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_0$, where $\Pi_{\mathcal{F}}$ is the generalized projection from E onto $\mathcal{F}$.

Proof. First, we show that C_n is closed and convex for each $n \ge 1$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_h is closed and convex for some h. For $z \in C_h$, we see that $\phi(z, y_h) \le \phi(z, x_h)$ is equivalent to

$$
2\langle z, J x_h - J y_h \rangle \le \|x_h\|^2 - \|y_h\|^2. \tag{2.2}
$$

It is easy to see that C_{h+1} is closed and convex. Then, for each $n \ge 1$, C_n is closed and convex. Now, we are in a position to show that $\mathcal{F} \subset C_n$ for each $n \ge 1$. Indeed, $\mathcal{F} \subset C_1 = C$ is obvious. Suppose that $\mathcal{F} \subset C_h$ for some h. Then, for all $w \in \mathcal{F} \subset C_h$, we have

$$
\begin{aligned}
\phi(w, z_h) &= \phi\Big(w, J^{-1}(\beta_{h,0} J x_h + \beta_{h,1} J T x_h + \beta_{h,2} J S x_h)\Big) \\
&= \|w\|^2 - 2\langle w, \beta_{h,0} J x_h + \beta_{h,1} J T x_h + \beta_{h,2} J S x_h \rangle \\
&\quad + \|\beta_{h,0} J x_h + \beta_{h,1} J T x_h + \beta_{h,2} J S x_h\|^2 \\
&\le \|w\|^2 - 2\beta_{h,0}\langle w, J x_h\rangle - 2\beta_{h,1}\langle w, J T x_h\rangle - 2\beta_{h,2}\langle w, J S x_h\rangle \\
&\quad + \beta_{h,0}\|x_h\|^2 + \beta_{h,1}\|T x_h\|^2 + \beta_{h,2}\|s x_h\|^2 \\
&= \beta_{h,0}\phi(w, x_h) + \beta_{h,1}\phi(w, T x_h) + \beta_{h,2}\phi(w, S x_h) \\
&\le \beta_{h,0}\phi(w, x_h) + \beta_{h,1}\phi(w, x_h) + \beta_{h,2}\phi(w, x_h) \\
&= \phi(w, x_h).
\end{aligned}
\tag{2.3}
$$

It follows that

$$\begin{aligned}
\phi(w, y_h) &= \phi\Big(w, J^{-1}(\alpha_h J x_h + (1-\alpha_h) J z_h)\Big) \\
&= \|w\|^2 - 2\langle w, \alpha_h J x_h + (1-\alpha_h) J z_h\rangle + \|\alpha_h J x_h + (1-\alpha_h) J z_h\|^2 \\
&\leq \|w\|^2 - 2\alpha_h\langle w, J x_h\rangle - 2(1-\alpha_h)\langle w, J z_h\rangle + \alpha_h\|x_h\|^2 + (1-\alpha_h)\|z_h\|^2 \\
&= \alpha_h \phi(w, x_h) + (1-\alpha_h)\phi(w, z_h) \\
&\leq \alpha_h \phi(w, x_h) + (1-\alpha_h)\phi(w, x_h) \\
&= \phi(w, x_h),
\end{aligned} \tag{2.4}$$

which shows that $w \in C_{h+1}$. This implies that $\mathcal{F} \subset C_n$ for each $n \geq 1$. On the other hand, we obtain from Lemma 1.4 that

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(w, x_0) - \phi(w, x_n) \leq \phi(w, \mathrm{x}_0), \tag{2.5}$$

for each $w \in \mathcal{F} \subset C_n$ and for each $n \geq 1$. This shows that the sequence $\phi(x_n, x_0)$ is bounded. From (1.6), we see that the sequence $\{x_n\}$ is also bounded. Since the space is reflexive, we may, without loss of generality, assume that $x_n \rightharpoonup \overline{x}$. Note that C_n is closed and convex for each $n \geq 1$. It is easy to see that $\overline{x} \in C_n$ for each $n \geq 1$. Note that

$$\phi(x_n, x_0) \leq \phi(\overline{x}, x_0). \tag{2.6}$$

It follows that

$$\phi(\overline{x}, x_0) \leq \liminf_{n\to\infty} \phi(x_n, x_0) \leq \limsup_{n\to\infty} \phi(x_n, x_0) \leq \phi(\overline{x}, x_0). \tag{2.7}$$

This implies that

$$\lim_{n\to\infty} \phi(x_n, x_0) = \phi(\overline{x}, x_0). \tag{2.8}$$

Hence, we have $\|x_n\| \to \|\overline{x}\|$ as $n \to \infty$. In view of the Kadec-Klee property of E, we obtain that $x_n \to \overline{x}$ as $n \to \infty$.

Next, we show that $\overline{x} \in F(T)$. By the construction of C_n, we have that $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_n$. It follows that

$$\begin{aligned}
\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0) \\
&\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\
&= \phi(x_{n+1}, x_0) - \phi(x_n, x_0).
\end{aligned} \tag{2.9}$$

Letting $n \to \infty$ in (2.9), we obtain that $\phi(x_{n+1}, x_n) \to 0$. In view of $x_{n+1} \in C_{n+1}$, we arrive at $\phi(x_{n+1}, y_n) \le \phi(x_{n+1}, x_n)$. It follows that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{2.10}$$

From (1.6), we can obtain that

$$\|y_n\| \longrightarrow \|\overline{x}\| \quad \text{as } n \longrightarrow \infty. \tag{2.11}$$

It follows that

$$\|Jy_n\| \longrightarrow \|J\overline{x}\| \quad \text{as } n \longrightarrow \infty. \tag{2.12}$$

This implies that $\{Jy_n\}$ is bounded. Note that E is reflexive and E^* is also reflexive. We may assume that $Jy_n \rightharpoonup x^* \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an $x \in E$ such that $Jx = x^*$. It follows that

$$\begin{aligned} \phi(x_{n+1}, y_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|y_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|Jy_n\|^2. \end{aligned} \tag{2.13}$$

Taking $\liminf_{n\to\infty}$, the both sides of equality above yield that

$$\begin{aligned} 0 &\ge \|\overline{x}\|^2 - 2\langle \overline{x}, x^*\rangle + \|x^*\|^2 \\ &= \|\overline{x}\|^2 - 2\langle \overline{x}, Jx\rangle + \|Jx\|^2 \\ &= \|\overline{x}\|^2 - 2\langle \overline{x}, Jx\rangle + \|x\|^2 \\ &= \phi(\overline{x}, x). \end{aligned} \tag{2.14}$$

That is, $\overline{x} = x$, which in turn implies that $x^* = J\overline{x}$. It follows that $Jy_n \rightharpoonup J\overline{x} \in E^*$.From (2.12) and since E^* enjoys the Kadec-Klee property, we obtain that

$$Jy_n - J\overline{x} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.15}$$

Note that $J^{-1} : E^* \to E$ is demicontinuous. It follows that $y_n \rightharpoonup \overline{x}$. From (2.11) and since E enjoys the Kadec-Klee property, we obtain that

$$y_n \longrightarrow \overline{x} \quad \text{as } n \longrightarrow \infty. \tag{2.16}$$

Note that

$$\|x_n - y_n\| \le \|x_n - \overline{x}\| + \|\overline{x} - y_n\|. \tag{2.17}$$

It follows that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{2.18}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{2.19}$$

On the other hand, we see from the definition of y_n that

$$\|Jy_n - Jx_n\| = (1-\alpha_n)\|Jz_n - Jx_n\|. \tag{2.20}$$

In view of the assumption on $\{\alpha_n\}$ and (2.19), we see that

$$\lim_{n\to\infty} \|Jx_n - Jz_n\| = 0. \tag{2.21}$$

On the other hand, since $J : E \to E^*$ is demicontinuous, we have $Jx_n \rightharpoonup J\overline{x} \in E^*$. In view of

$$|\|Jx_n\| - \|J\overline{x}\|| = |\|x_n\| - \|\overline{x}\|| \leq \|x_n - \overline{x}\|, \tag{2.22}$$

we arrive at $\|Jx_n\| \to \|J\overline{x}\|$ as $n \to \infty$. By virtue of the Kadec-Klee property of E^*, we obtain that $\|Jx_n - J\overline{x}\| \to 0$ as $n \to \infty$. Note that

$$\|Jz_n - J\overline{x}\| \leq \|Jz_n - Jx_n\| + \|Jx_n - J\overline{x}\|. \tag{2.23}$$

In view of (2.21), we arrive at $\lim_{n\to\infty}\|Jz_n - J\overline{x}\| = 0$. Since $J^{-1} : E^* \to E$ is demicontinuous, we have $z_n \rightharpoonup \overline{x}$. Note that

$$|\|z_n\| - \|x_n\|| = |\|Jz_n\| - \|J\overline{x}\|| \leq \|Jz_n - J\overline{x}\|. \tag{2.24}$$

It follows that $\|z_n\| \to \|\overline{x}\|$ as $n \to \infty$. Since E enjoys the Kadec-Klee property, we obtain that $\lim_{n\to\infty}\|z_n - \overline{x}\| = 0$. Note that

$$\|z_n - x_n\| \leq \|z_n - \overline{x}\| + \|\overline{x} - x_n\|. \tag{2.25}$$

It follows that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{2.26}$$

Let $r = \max\{\sup_{n\geq 1}\{\|x_n\|\}, \sup_{n\geq 1}\{\|Tx_n\|\}, \sup_{n\geq 1}\{\|Sx_n\|\}\}$. Fixing $q \in \mathcal{F}$, we have from Lemma 1.6 that

$$\begin{aligned}
\phi(q, z_n) &= \phi\Big(q, J^{-1}(\beta_{n,0}Jx_n + \beta_{n,1}JTx_n + \beta_{n,2}JSx_n)\Big) \\
&= \|q\|^2 - 2\langle q, \beta_{n,0}Jx_n + \beta_{n,1}JTx_n + \beta_{n,2}JSx_n)\rangle \\
&\quad + \|\beta_{n,0}Jx_n + \beta_{n,1}JTx_n + \beta_{n,2}JSx_n\|^2 \\
&\leq \|q\|^2 - 2\beta_{n,0}\langle q, Jx_n\rangle - 2\beta_{n,1}\langle q, JTx_n\rangle - 2\beta_{n,2}\langle q, JSx_n\rangle \\
&\quad + \beta_{n,0}\|Jx_n\|^2 + \beta_{n,1}\|JTx_n\|^2 + \beta_{n,2}\|JSx_n\|^2 - \beta_{n,0}\beta_{n,1}g(\|Jx_n - JTx_n\|) \\
&= \beta_{n,0}\phi(q, x_n) + \beta_{n,1}\phi(q, Tx_n) + \beta_{n,2}\phi(q, Sx_n) - \beta_{n,0}\beta_{n,1}g(\|Jx_n - JTx_n\|) \\
&\leq \beta_{n,0}\phi(q, x_n) + \beta_{n,1}\phi(q, x_n) + \beta_{n,2}\phi(q, x_n) - \beta_{n,0}\beta_{n,1}g(\|Jx_n - JTx_n\|) \\
&= \phi(q, x_n) - \beta_{n,0}\beta_{n,1}g(\|Jx_n - JTx_n\|).
\end{aligned} \tag{2.27}$$

It follows that

$$\beta_{n,0}\beta_{n,1}g(\|Jx_n - JTx_n\|) \leq \phi(q, x_n) - \phi(q, z_n). \tag{2.28}$$

On the other hand, we have

$$\begin{aligned}
\phi(q, x_n) - \phi(q, z_n) &= \|x_n\|^2 - \|z_n\|^2 - 2\langle q, Jx_n - Jz_n\rangle \\
&\leq \|x_n - z_n\|(\|x_n\| + \|z_n\|) + 2\|q\|\|Jx_n - Jz_n\|.
\end{aligned} \tag{2.29}$$

It follows from (2.21) and (2.26) that

$$\lim_{n\to\infty}(\phi(q, x_n) - \phi(q, z_n)) = 0. \tag{2.30}$$

In view of (2.28) and the assumption $\liminf_{n\to\infty}\beta_{n,0}\beta_{n,1} > 0$, we see that

$$\lim_{n\to\infty} g(\|Jx_n - JTx_n\|) = 0. \tag{2.31}$$

It follows from the property of g that

$$\lim_{n\to\infty}\|Jx_n - JTx_n\| = 0. \tag{2.32}$$

Note that

$$\lim_{n\to\infty}\|Jx_n - J\overline{x}\| = 0. \tag{2.33}$$

On the other hand, we have

$$\|JTx_n - J\overline{x}\| \le \|JTx_n - Jx_n\| + \|Jx_n - J\overline{x}\|. \tag{2.34}$$

From (2.32) and (2.33), we arrive at

$$\lim_{n\to\infty} \|JTx_n - J\overline{x}\| = 0. \tag{2.35}$$

Note that $J^{-1} : E^* \to E$ is demicontinuous. It follows that $Tx_n \rightharpoonup \overline{x}$. On the other hand, we have

$$|\|Tx_n\| - \|\overline{x}\|| = |\|JTx_n\| - \|J\overline{x}\|| \le \|JTx_n - J\overline{x}\|. \tag{2.36}$$

In view of (2.35), we obtain that $\|Tx_n\| \to \|\overline{x}\|$ as $n \to \infty$. Since E enjoys the Kadec-Klee property, we obtain that

$$\lim_{n\to\infty} \|Tx_n - \overline{x}\| = 0. \tag{2.37}$$

It follows from the closedness of T_1 that $T\overline{x} = \overline{x}$. By repeating (2.27)–(2.37), we can obtain that $\overline{x} \in F(S)$. This shows that $\overline{x} \in \mathcal{F}$.

Finally, we show that $\overline{x} = \Pi_{\mathcal{F}} x_0$. From $x_n = \Pi_{C_n} x_0$, we have

$$\langle x_n - w, Jx_0 - Jx_n\rangle \ge 0, \quad \forall w \in \mathcal{F} \subset C_n. \tag{2.38}$$

Taking the limit as $n \to \infty$ in (2.38), we obtain that

$$\langle \overline{x} - w, Jx_0 - J\overline{x}\rangle \ge 0, \quad \forall w \in \mathcal{F}, \tag{2.39}$$

and hence $\overline{x} = \Pi_{F(T)} x_0$ by Lemma 1.3. This completes the proof. □

Remark 2.2. Theorem 2.1 improves Theorem SWX in the following aspects:

(a) from the point of view on computation, we remove the set "Q_n" in Theorem SWX;

(b) from the point of view on the framework of spaces, we extend Theorem SWX from a uniformly smooth and uniformly convex Banach space to a uniformly smooth and strictly convex Banach space which enjoys the Kadec-Klee property. Note that every uniformly convex Banach space enjoys the Kadec-Klee property.

If $\alpha_n = 0$ for each $n \geq 0$, then Theorem 2.1 is reduced to the following.

Corollary 2.3. *Let E be a uniformly smooth and strictly convex Banach space which enjoys the Kadec-Klee property and C a nonempty closed and convex subset of E. Let $T : C \to C$ and $S : C \to C$ be two closed and hemirelatively nonexpansive mappings such that $\mathcal{F} = F(T) \cap F(S)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered}
x_0 \in E \text{ chosen arbitrarily}, \\
C_1 = C, \\
x_1 = \Pi_{C_1} x_0, \\
y_n = J^{-1}\left(\beta_{n,0} J x_n + \beta_{n,1} J T x_n + \beta_{n,2} J S x_n\right), \\
C_{n+1} = \left\{z \in C_n : \phi\left(z, y_n\right) \leq \phi\left(z, x_n\right)\right\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0,
\end{gathered} \tag{2.40}$$

where $\{\beta_{n,0}\}$, $\{\beta_{n,1}\}$, and $\{\beta_{n,2}\}$ are real sequences in $[0,1]$ satisfying the following restrictions:

(a) $\beta_{n,0} + \beta_{n,1} + \beta_{n,2} = 1$;

(b) $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,1} > 0$ *and* $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,2} > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_0$, where $\Pi_{\mathcal{F}}$ is the generalized projection from E onto $\mathcal{F}$.

If $T = S$, then Corollary 2.3 is reduced to the following.

Corollary 2.4. *Let E be a uniformly smooth and strictly convex Banach space which enjoys the Kadec-Klee property and C a nonempty closed and convex subset of E. Let $T : C \to C$ be a closed and hemirelatively nonexpansive mapping with a nonempty fixed point set. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered}
x_0 \in E \text{ chosen arbitrarily}, \\
C_1 = C, \\
x_1 = \Pi_{C_1} x_0, \\
y_n = J^{-1}\left(\beta_n J x_n + \left(1 - \beta_n\right) J T x_n\right), \\
C_{n+1} = \left\{z \in C_n : \phi\left(z, y_n\right) \leq \phi\left(z, x_n\right)\right\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0,
\end{gathered} \tag{2.41}$$

where $\{\beta_n\}$ is a real sequence in $[0,1]$ satisfying $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$. Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_0$, where $\Pi_{\mathcal{F}}$ is the generalized projection from E onto $\mathcal{F}$.

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